G. Spontaneous Emission

Einstein: $A_{21} = \frac{\hbar\omega^3}{\pi^2 c^3} B_{21}$; Schrödinger's QM $\lambda_{1\to2} = \underbrace{\frac{\pi e^2}{3\varepsilon_0\hbar^2}|r_{21}|^2}_{B_{21}} \cdot U(\omega_{21})$

↑ Schrödinger's QM can't treat vacuum

↑ Einstein didn't know

Putting the results together

$$\boxed{A_{21} = \frac{\omega^3}{3\pi\varepsilon_0 c^3\hbar} e^2|r_{21}|^2} \quad (32)$$

a QM result (without effort!)

↪ Units: $\frac{1}{\text{time}}$

$\frac{1}{A}$ is a time

$\tau = \frac{1}{A}$ = life time of an excited state

$$= \frac{3\pi\varepsilon_0 c^3\hbar}{\omega^3} \frac{1}{e^2|r_{21}|^2}$$

- Let $N_2(t)$ be number of excited atoms (at state "2") at time t

$N_2(t)$ drops due to spontaneous emission (NO Stimulation)

Recall: $A_{(21)}$ OR A is prob. per atom per unit time to decay to a lower state "1"

$$\therefore \frac{dN_2(t)}{dt} = -A N_2(t)$$

↖ Einstein's A coefficient

$$\Rightarrow N_2(t) = N_2(0)\, e^{-At} \equiv N_2(0)\, e^{-t/\tau} \quad (\tau = \text{life time})$$

$$\therefore \boxed{\tau = \frac{1}{A}} \quad (33)$$

Einstein's A-coefficient inversed is excited state's lifetime

(really getting many things from nothing!)

For state "2" having only state "1" to decay to, A_{21} is given by (32)

Thus
$$\boxed{\tau = \frac{3\pi\varepsilon_0 c^3 \hbar}{\omega^3}\frac{1}{e^2|r_{21}|^2}} \quad (34)$$
(Magical!)

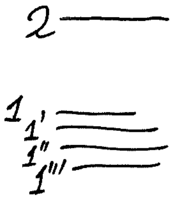
Appreciation: QM provides a formula to calculate τ through evaluating an integral r_{21} (recall $\omega = \frac{E_2 - E_1}{\hbar}$)

- $\sim \frac{1}{\omega^3} \Rightarrow$ highly excited states have short τ

  Very hard to study! (A hot research topic, see "Rydberg Atoms")
- state with lower state (allowed) to go $\Rightarrow |r_{21}|^2$ not small

  $\Rightarrow$ short τ
- state with no lower state to go $\Rightarrow$ long τ (meta stable state)

What if State "2" has many lower states to go?



$$\frac{dN_2(t)}{dt} = -A_{21}N_2(t) - A_{21'}N_2(t) - A_{21''}N_2(t) \cdots$$

$$\therefore \quad \tau_2 = \frac{1}{A_{21} + A_{21'} + A_{21''} + A_{21'''} + \cdots}$$

this is τ of state 2

- More states to go $\Rightarrow$ shorter τ
- Biggest A dominates

OR

$$\frac{1}{\tau_2} = A_{21} + A_{21'} + A_{21''} + A_{21'''} + \cdots$$

$$= \frac{1}{\tau_{21}} + \frac{1}{\tau_{21'}} + \frac{1}{\tau_{21''}} + \frac{1}{\tau_{21'''}} + \cdots$$

due to 2→1 due to 2→1' due to 2→1''' (each can be handled by QM)

Key Point: life time is related to spontaneous emission through the A-coefficient(s)

Lifetime of Excited States τ

- Excited state could have short lifetime τ because
  - there is (are) lower state(s) with allowed transitions ($\because |r_{21}|^2$)
  - excited state is high in energy ($\because \omega^3$)
  - many lower states to go to

Short lifetime	Long Lifetime
$\tau \sim 10^{-9}\,s - 10^{-8}\,s$	$\tau \sim \underbrace{10^{-3}\,s}$
H-atom: $2p \rightarrow 1s$ ($\sim \underbrace{10^{-9}\,s}$)	[Metastable state] (important for operation of LASER)

$$\boxed{\text{Life time is related to spontaneous emission}}$$